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Journal of Functional Analysis 210 (2004) 43–72

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Functional calculus under the Tadmor–Ritt condition, and free interpolation by polynomials of a given degree

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Received 12 February 2003; accepted 14 August 2003

Communicated by D. Sarason

Abstract

For Banach space operators T satisfying the Tadmor–Ritt condition $\|(zI - T)^{-1}\| \leqslant C|z-1|^{-1}$, $|z|>1$, we prove that the best-possible constant $C_T(n)$ bounding the polynomial calculus for T, $\|p(T)\| \leqslant C_T(n)\|p\|_\infty$, $\deg(p) \leqslant n$, behaves (in the worst case) as $\log n$ as $n \to \infty$. This result is based on a new free (Carleson type) interpolation theorem for polynomials of a given degree.

Keywords: Tadmor and Ritt conditions; Power bounded; Functional calculus; Cauchy–Stieltjes integrals; Multipliers; Carleson type free interpolation by polynomials; Basis and unconditional basis constants

Notation

$\mathbb{D}$ denotes the open-unit disc of the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. H^∞ is the Banach algebra of bounded analytic functions on $\mathbb{D}$ equipped with the supremum norm $\|.\|_\infty$.

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2003.08.002

Introduction

In this paper, we consider Banach space operators $T: X \to X$ satisfying the following Tadmor–Ritt condition (TR):

$$||R_z(T)|| \leqslant \frac{C}{|z-1|} \quad \text{for } z \in \mathbb{C}, |z| > 1, \tag{TR}$$

where $R_z(T) = R_z = (zI - T)^{-1}$ is the resolvent of T. Conditions of this type (see also the Ritt condition (R) below) appear in numerical analysis, for matrices instead of operators but of unboundedly varying size, when an initial value problem $f' = Af + g$ is replaced by a suitable Runge–Kutta iterative scheme $u_n = Tu_{n-1} + r_n$, where $T = \varphi(A)$ is a relevant expression in the generator A. See [BDS00,Nev93] for more details and references. The main objective is to prove that the above scheme is stable, that is

$$\sup_{n \geqslant 0} ||T^n|| = PB(T) < \infty,$$

and to find a numerical bound for $PB(T)$ in terms of $C = C(T)$. The problem is raised in Ritt [Rit53], and after a long evolution, Lyubich [Lyu99] and Nagy and Zemanek [NZ99] derived from a preceding result of Nevanlinna [Nev93] that, indeed, $PB(T) < \infty$. Since these results, considerable efforts were made by many people to obtain a sharp bound for $PB(T)$. We refer to [EFR02] for the best-known (2002) estimate, $PB(T) \leqslant C^2$, as well as for more references.

Condition (TR) is often referred to Ritt [Rit53] but, in fact, it was introduced by Tadmor [Tad86]. The original Ritt condition (R) is weaker and consists of the following properties of a Banach space operator T:

$$\begin{gathered} \sigma(T) \subset \mathbb{D} \cup \{1\}, \\ ||R_z|| \leqslant \frac{L}{|z-1|} \quad \text{for } |z| > 1, 0 < |z-1| < \varepsilon, \end{gathered} \tag{R}$$

where L and ε are some constants. Obviously (TR) implies (R) with $L = C$. Conversely, by the maximum modulus principle, (R) implies (TR) but with a constant $C > 0$ which can be much bigger than L (and, in fact, is not controllable in terms of L). Therefore, having in mind possible applications to numerical analysis, one should distinguish conditions (TR) and (R). We refer to [BDS00] for a discussion and to [Vitb] for an estimate of $\limsup_n ||T^n||$ in terms of L.

In this paper, we apply a duality approach already used in [Vita] to estimate the functional calculus $f \mapsto f(T)$ of an operator satisfying the Tadmor–Ritt condition.

First, we reduce the problem to certain estimates through Cauchy–Stieltjes integrals and their multipliers (Theorem 1.2 and Corollary 1.3). It is worth mentioning that a slight modification of this estimate and a classical Riesz theorem ([Rie10], see also [Lan46]), can give a simple proof to the (known) fact that $(R) \Rightarrow \sup_{n \geqslant 0} ||T^n|| < \infty$ and $\sup_{n \geqslant 0} n||T^n - T^{n+1}|| < \infty$. See [Vitb] for details.

Then, we deal with the polynomial calculus for operators satisfying condition (TR). The following estimate was already proved in the Hilbert space case under the weaker condition that T is power-bounded, Peller [Pel82], see also Wojtaszczyk [Woj91, Corollary III.F.18] (for a different proof due to Pisier). We extend it to the Banach space case under the (TR) condition, as follows (note that it is no longer true if only the power boundedness is assumed).

(1) $\|p(T)\| \leqslant a_1 C \log(en)\|p\|_\infty$ for every polynomial of degree $\leqslant n$, where $a_1 > 0$ is an absolute constant and $n \geqslant 1$.

Then we show that the preceding estimate is sharp, namely

(2) there exist a constant $a_2 > 0$ and an operator $T \in (TR)$ such that

$$C_T(n) = \max\{\|p(T)\| : \deg(p) \leqslant n, \|p\|_\infty \leqslant 1\} \geqslant a_2 \log(en)$$

for every $n \geqslant 1$.

In order to construct an operator T required by (2), we prove the following free interpolation theorem for polynomials of a given degree.

(3) there exist absolute constants $a, b > 0$ such that given an interpolating sequence $(\lambda_k)_{k \geqslant 1}$ ordered as $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$ and having the Carleson constant $\delta > 0$, where

$$\delta = \inf_{k \geqslant 1} \prod_{j \neq k} |b_{\lambda_j}(\lambda_k)|,$$

and a sequence $(a_k)_{k \geqslant 1}$ such that $a_k \in \mathbb{C}$, $|a_k| \leqslant 1$, there exists a polynomial p of degree $\leqslant n$ such that

(a) $p(\lambda_k) = a_k,\ k = 1, \dots, m(n)$,
(b) $\|p\|_\infty \leqslant \frac{b}{\delta^{32}}$,

where $m(n) \geqslant a\, card\{\lambda_k : |\lambda_k| \leqslant 1 - \frac{1}{n}\}$. The quantity $card\{\lambda_k : |\lambda_k| \leqslant 1 - \frac{1}{n}\}$ depends on the sequence $(\lambda_k)_{k \geqslant 1}$ but it is always $O(n)$ as $n \to \infty$. The above bound for $m(n)$ cannot be essentially improved, namely, for every interpolating sequence $(\lambda_k)_{k \geqslant 1}$, $0 < \lambda_k < 1$, there exists a constant $c > 0$ such that (a) and (b) imply $m(n) \leqslant c\, card\{\lambda_k : |\lambda_k| \leqslant 1 - \frac{1}{n}\}$. One can also compare our estimates with a result of Bourgain [Bou86] who proved (in our notation) that the above interpolation is possible for $m(n) = o(n^{\frac{1}{4}})$ points as $n \to \infty$. For more comments see the remarks after Theorem 2.8.

It is more difficult to find an operator T satisfying properties of point (2) and acting on a given Banach space X because our construction depends on the existence of a basis $\mathscr{E} = (e_j)_{j \geqslant 1}$ having the maximal growth of the unconditional basis constant $UB((e_j)_{j=1}^n) = UB(\mathscr{E}_n)$, namely $UB(\mathscr{E}_n) \geqslant an$, where $a > 0$ is a constant. We exhibit such bases for the spaces l^1 and c_0, and hence we are able to construct an operator satisfying (2) on every Banach space X containing a complemented subspace isomorphic to one of these spaces. (For example, on $X = L^1(\mu)$, $L^\infty(\mu)$ for every measure μ not reduced to a finite sum of δ-measures, or on $\mathscr{C}(K)$, where K is an infinite compact.) On a Hilbert space, using results from [STW03], we can prove the following.

(4) For every $\varepsilon > 0$, there exists an operator $T = T_\varepsilon \in (TR)$ on a Hilbert space H, $\dim H = \infty$, such that

$$C_T(n) \geqslant a_2(\log(en))^{1-\varepsilon},$$

for every $n \geqslant 1$, where $a_2 = a_2(\varepsilon) > 0$. One can make ε tend to 0 but the corresponding (TR) constant $C(T_\varepsilon)$ tends to infinity and $a_2(\varepsilon) \to 0$.

In order to explain better this latter dependence of $C_{T_\varepsilon}(n)$ from $C(T_\varepsilon)$, we introduce the following quantity. Given a Banach space X and an integer $n \geqslant 1$, we set

$$C(X,n) = \sup\left\{\frac{C_T(n)}{C(T)} : T : X \to X \text{ satisfying } (TR)\right\}.$$

Then, we have the following (see Section 3 below for details and, in particular, for geometric terminology).

(5) For a Banach space X containing uniformly and uniformly complemented copies of l_n^1 or l_n^∞ (as the spaces $L^1(\mu)$, $L^\infty(\mu)$, $\mathscr{C}(K)$ having infinite dimension), there exist constants a, $b > 0$ such that

$$a \log(en) \leqslant C(X,n) \leqslant b \log(en)$$

for every $n \geqslant 1$.

For an infinite dimensional Hilbert space $X = H$, as well as for any Banach space X of type > 1 (as the spaces $L^p(\mu)$, $1 < p < \infty$), there exist constants a, $b > 0$ such that

$$a \frac{\log(en)}{(\log\log(en))^{\frac{3}{2}}} \leqslant C(X,n) \leqslant b \log(en)$$

for every $n \geqslant 2$.

Also, there exist constants a, $b > 0$ such that for every Banach space X, $\dim X = \infty$, we have

$$a \frac{(\log(en))^{\frac{1}{2}}}{(\log\log(en))^{\frac{3}{2}}} \leqslant C(X,n) \leqslant b \log(en)$$

for every $n \geqslant 1$.

In general, for a given Banach space X, we do not know whether one of the above bounds is sharp. For more comments, see Section 3.

It is also worth mentioning that our approach permits to strengthen an example of Le Merdy [LM98] by giving an operator on a Hilbert space satisfying (TR) and not polynomially bounded not only over the unit disc but over an arbitrary "collar type" domain Col such that $\mathbb{D} \subset Col$ and $(1, \infty) \subset \mathbb{C} \backslash \overline{Col}$; see Fig. 1 in Section 2.

Yet another corollary of our estimates is the following polynomial version of a known result on the behavior of the derivative of an H^∞ function (see for instance [Nik02, Vol. 2, p.376] for references and a discussion).

(6) Let

$$V_n = \sup\left\{ \int_0^1 |f'(t)| dt : ||f||_\infty \leqslant 1, \ \deg(f) \leqslant n \right\}.$$

There exist constants a, $b > 0$ such that

$$a \log n \leqslant V_n \leqslant b \log n.$$

In conclusion, we compare our results with known estimates for operators from two other classes closely related to the class of Tadmor–Ritt operators (TR). Namely, let (PB) be the class of power bounded operators and (K) the class of operators satisfying the Kreiss condition (K):

$$||R_z(T)|| \leqslant \frac{K(T)}{|z| - 1} \quad \text{for } |z| > 1. \tag{K}$$

Then we have $(TR) \subset (PB) \subset (K)$, where the first written embedding is mentioned above, and the second one is an obvious norm one inclusion.

It is clear that $T \in (PB)$ implies

$$||f(T)|| \leqslant PB(T) \sum_{k \geqslant 0} |\hat{f}(k)|$$

for every $f = \sum_{k \geqslant 0} \hat{f}(k) z^k$ having $\sum_{k \geqslant 0} |\hat{f}(k)| = ||f||_{l^1_A} < \infty$. Thus, $||p(T)|| \leqslant PB(T)\sqrt{n+1}||p||_\infty$ for every polynomial p of $\deg(p) \leqslant n$ (by the Cauchy inequality), and hence

$$C_T(n) \leqslant PB(T)\sqrt{n+1}.$$

Conversely, there exist Banach space operators T (for instance, the shift $T(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$ on the spaces l^1 or c_0) such that

$$C_T(n) \geqslant a\sqrt{n+1},$$

where $a > 0$ is an absolute constant (for the shift operator T, the latter inequality follows when considering the Rudin–Shapiro polynomials p_k, $\deg(p_k) = 2^k$, $k = 0, 1, \ldots$, satisfying $||p_k||_{l^1_A} \geqslant 2^{\frac{k}{2}} ||p_k||_\infty$, see [GM79]). For operators $T \in (PB)$ acting on specific Banach spaces, the constants $C_T(n)$ can behave differently. For instance, as is already mentioned, for a Hilbert space $X = H$, it is shown by Peller [Pel82] that $C_T(n) \leqslant const(\log(n+1)e)$ (see also [Woj91, pp. 215–216], where it is proved that one can take $const = PB(T)^2$). In [Woj91], the problem of the sharpness of this estimate is raised, and an example of $T \in (PB)$ on a Hilbert space is exhibited such that $C_T(n) \geqslant a(\log(n+1))^{\frac{1}{2}}$. From our results, we infer operators $T_\varepsilon \in (TR) \subset (PB)$ with $C_{T_\varepsilon}(n) \geqslant a_\varepsilon(\log(n+1))^{1-\varepsilon}$, for an arbitrary $\varepsilon > 0$, see (4) and (5) above.

For the largest class of Kreiss operators, it is known that $T \in (K)$ implies

$$C_T(n) \leqslant bK(T)(n+1),$$

where $b > 0$ is an absolute constant (see Vitse [Vita], and also Peller [Pel84], where a general functional calculus is constructed without specifying consequences for polynomials of a given degree). Moreover, there exist Banach space operators T satisfying (K) and such that

$$C_T(n) \geqslant ||T^n|| \geqslant a(n+1),$$

where $a > 0$ is an absolute constant, see Vitse [Vita]. Upper estimates for the polynomial functional calculus of Kreiss operators on specific Banach spaces, which, in principle, could be better than the general one, are not known (for instance, for Hilbert spaces). As to the lower ones, Spijker et al. [STW03] constructed operators T_ε, $\varepsilon > 0$, on a Hilbert space, satisfying (K) and such that $C_T(n) \geqslant ||T^n|| \geqslant a_\varepsilon(n+1)^{1-\varepsilon}$ (where $a_\varepsilon \to 0$ as $\varepsilon \to 0$). See also Remark 2.13.

The paper is organized as follows. In Section 1, we write down explicitly the duality adapted to the Tadmor–Ritt type conditions and give some consequences, in particular, a multiplier estimate for $||f(T)||$. Section 2 is devoted to results described in (1)–(3) and (6) of the above list, and Section 3 to results (4)–(5).

1. Cauchy–Stieltjes integrals and their multipliers

Let $\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})$ be the space of all functions continuous on $\mathbb{C}\backslash\mathbb{D}$, holomorphic on $\mathbb{C}\backslash\overline{\mathbb{D}}$ and vanishing at ∞. It is endowed with the supremum norm $||g||_\infty = \sup\{|g(z)| : z \in \mathbb{C}\backslash\mathbb{D}\}$. $\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})$ is a closed subspace of the space $\mathscr{C}(\mathbb{T})$ of all continuous functions on the unit circle $\mathbb{T} = \partial\mathbb{D}$. Rational functions are dense in $\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})$, that is

$$\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}}) = span_{\mathscr{C}(\mathbb{T})}\left\{\frac{1}{z-\zeta} : \zeta \in \mathbb{D}\right\},$$

where $span_{\mathscr{C}(\mathbb{T})}$ stands for the closed linear hull in the space $\mathscr{C}(\mathbb{T})$. Also

$$\mathscr{C}(\mathbb{T}) = span_{\mathscr{C}(\mathbb{T})}\left\{\frac{1}{z-\zeta} : \zeta \in \mathbb{C}\backslash\mathbb{T}\right\}.$$

The dual space of $\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})$ is

$$\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})^* = \mathscr{M}(\mathbb{T})/\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})^\perp$$

with respect to the standard duality between $\mathscr{C}(\mathbb{T})$ and $\mathscr{M}(\mathbb{T})$,

$$(f, \mu) = \int_{\mathbb{T}} f \, d\mu,$$

where $\mathscr{M}(\mathbb{T})$ stands for the space of complex Borel measures on $\mathbb{T}$, and

$$\begin{aligned}\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})^{\perp} &= \{\mu \in \mathscr{M}(\mathbb{T}) : (f, \mu) = 0 \text{ for every } f \in \mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})\} \\ &= \left\{\mu \in \mathscr{M}(\mathbb{T}) : \int_{\mathbb{T}} \frac{d\mu(z)}{z - \zeta} = 0 \text{ for every } \zeta \in \mathbb{D}\right\}.\end{aligned}$$

Given a measure $\mu \in \mathscr{M}(\mathbb{T})$, we denote by C^{μ} a Cauchy–Stieltjes integral

$$C^{\mu}(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{\zeta - z}, \ z \in \mathbb{C}\backslash\mathbb{T}.$$

By previous remarks the mapping $\mu \mapsto C^{\mu}$, $\mu \in \mathscr{M}(\mathbb{T})$, is injective, and by the theorem of the Brothers Riesz $\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})^{\perp}$ can be identified with the Hardy class H^1 (see [Dur70]). Therefore, the dual space $\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})^{*}$ can be identified with the space of Cauchy–Stieltjes integrals on $\mathbb{D}$,

$$CSI(\mathbb{D}) = \{f \in Hol(\mathbb{D}) : f = C^{\mu} \text{ for some } \mu \in \mathscr{M}(\mathbb{T})\}$$

endowed with the quotient norm

$$||f||_{CSI(\mathbb{D})} = \inf\{||\mu|| : f = C^{\mu}\} = ||\mu||_{\mathscr{M}(\mathbb{T})/H^1}.$$

In the case when $f \in H^1$, we have $f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)\, d\zeta}{\zeta - z}$, $z \in \mathbb{D}$, and hence

$$||f||_{CSI(\mathbb{D})} = \left|\left|\frac{1}{2\pi i} f d\zeta\right|\right|_{\mathscr{M}(\mathbb{T})/H^1}.$$

Another special case is $f = C^{\mu}$, where μ is a measure on $\mathbb{T}$ singular with respect to the arc length. In this case the theorem of the Brothers Riesz implies that $||f||_{CSI(\mathbb{D})} = ||\mu||_{\mathscr{M}(\mathbb{T})}$. It follows that for an arbitrary $\mu \in \mathscr{M}(\mathbb{T})$, $||C^{\mu}||_{CSI(\mathbb{D})} = ||C^{\mu_a} + C^{\mu_s}||_{CSI(\mathbb{D})} = ||C^{\mu_a}||_{CSI(\mathbb{D})} + ||C^{\mu_s}||_{CSI(\mathbb{D})}$, where $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to the arc length. The duality form between $\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})$ and $CSI(\mathbb{D})$ is

$$\langle g, C^{\mu} \rangle = (g, \mu) = \int_{\mathbb{T}} g \, d\mu.$$

Now, let $f = \sum_{k \geqslant 0} a_k z^k$ be the Taylor decomposition of a function $f \in CSI(\mathbb{D})$ and $g \in \mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})$, $g = \sum_{k \geqslant 0} \frac{b_k}{z^{k+1}}$. Then, it is easy to see that if one of these two series uniformly converges on $\mathbb{T}$, we get

$$\langle g,f\rangle = \sum\nolimits_{k\geqslant 0} a_k b_k.$$

For many other cases we can regularize the series convergence giving sense to the equality $\langle g,f\rangle = \sum_{k\geqslant 0} a_k b_k$.

The above duality can be written in several other ways. One of the equivalent forms is the following. We realize the duality $\mathscr{C}(\mathbb{T})^* = \mathscr{M}$ by the sesquilinear form $(f,\mu) = \int_{\mathbb{T}} f\, d\overline{\mu}$, where $\overline{\mu}(\sigma) = \overline{\mu(\sigma)}$, $\sigma\subset\mathbb{T}$, is the measure complex-conjugate to $\mu\in\mathscr{M}$. It follows that $\mathscr{C}_A(\mathbb{D})^* = \mathscr{M}/\mathscr{C}_A^{\perp} = \mathscr{M}/H^1_-(\mathbb{D}) = CSI(\mathbb{D})$ with the duality $(f,g) = \sum_{n\geqslant 0}\hat{f}(n)\overline{\hat{g}(n)}$ for $f\in\mathscr{C}_A(\mathbb{D})$, $g\in CSI(\mathbb{D})$.

Finally, we mention that if $f\in Hol(\overline{\mathbb{D}})$ then

$$\frac{f}{1-z} = \frac{f(1)}{1-z} + \frac{f-f(1)}{1-z}\in CSI(\mathbb{D}).$$

Moreover,

$$\left\langle g,\frac{f}{1-z}\right\rangle = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(z)}{1-z}g(z)\,dz$$

for every $g\in\mathscr{C}_A(\mathbb{C}\backslash\overline{\mathbb{D}})$ having $g(1)=0$. Indeed,

$$\begin{aligned}\left\langle g,\frac{f}{1-z}\right\rangle &= f(1)\left\langle g,\frac{1}{1-z}\right\rangle + \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(z)-f(1)}{1-z}g(z)\,dz\\ &= f(1)g(1) + \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(z)-f(1)}{1-z}g(z)\,dz = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(z)}{1-z}g(z)\,dz,\end{aligned}$$

since $\int_{\mathbb{T}}\frac{g(z)}{1-z}\,dz = 0$ by the Cauchy theorem.

Now, we apply this duality to the functional calculus for a Tadmor–Ritt operator. We write $T\in(TR)_C$ for an operator satisfying the (TR) condition with a constant C.

Lemma 1.1. *If $T\in(TR_C)$ such that $1\notin\sigma(T)$, $f\in Hol(\overline{\mathbb{D}})$, and $x\in X$, $y\in X^*$. Then,*

$$(f(T)x,y) = \left\langle\frac{zf}{1-z},g\right\rangle = f(1)(x,y) + \left\langle\frac{f-f(1)}{1-z}, zg+(x,y)\right\rangle,$$

where $g = g_{x,y}(z) = \frac{(1-z)(R_z x,y)}{z}$ for $z\in\mathbb{C}\backslash\mathbb{D}$. Hence,

$$||f(T)||\leqslant(C+1)\left|\left|\frac{f}{1-z}\right|\right|_{CSI(\mathbb{D})}.$$

Proof. The function $z \mapsto R_z$ is holomorphic in $\mathbb{C}\backslash\mathbb{D}$ and we have

$$(f(T)x, y) = \frac{1}{2\pi i}\int_{\mathbb{T}} f(z)(R_z x, y)\,dz = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{zf(z)}{1-z} g(z)\,dz = \left\langle \frac{zf}{1-z}, g\right\rangle$$

(see above comments on the duality). Writing $f(z) = f(1) + (f(z) - f(1))$, we obtain the claimed formula. Therefore, using $|1-z|\,\|R_z\| \leqslant C$ for $z \in \mathbb{T}$, we get

$$|(f(T)x, y)| \leqslant \left(|f(1)| + (C+1)\left\|\frac{f - f(1)}{1-z}\right\|_{CSI(\mathbb{D})}\right)\cdot \|x\|\cdot\|y\|,$$

which implies

$$\begin{aligned}\|f(T)\| &\leqslant (C+1)\left(|f(1)| + \left\|\frac{f-f(1)}{1-z}\right\|_{CSI(\mathbb{D})}\right)\\ &= (C+1)\left(\left\|\frac{f(1)}{1-z}\right\|_{CSI(\mathbb{D})} + \left\|\frac{f-f(1)}{1-z}\right\|_{CSI(\mathbb{D})}\right)\\ &= (C+1)\left\|\frac{f}{1-z}\right\|_{CSI(\mathbb{D})}.\end{aligned}$$

The last equality holds because the measure corresponding to $\frac{f-f(1)}{1-z}$ is absolutely continuous with respect to the Lebesgue measure $|dz|$ whereas the measure corresponding to $\frac{f(1)}{z-1}$ is singular. $\square$

Theorem 1.2. *Let $T \in (TR_C)$ and $f \in Hol(\overline{\mathbb{D}})$. Then*

$$\|f(T)\| \leqslant (C+1)\left\|\frac{f}{1-z}\right\|_{CSI(\mathbb{D})}.$$

Proof. We apply the preceding lemma to rT, $0<r<1$, instead of T. Since

$$\|R_z(rT)\| = \frac{1}{r}\|R_{\frac{z}{r}}\| \leqslant \frac{C}{|z-r|} \leqslant \frac{C}{|z-1|}\cdot\frac{2}{1+r}$$

for every z, $|z|>1$, the operator rT satisfies the (TR) condition with the constant $C\frac{2}{1+r}$. Moreover, by the Riesz–Dunford calculus, we have $\lim_{r\to 1}\|f(rT) - f(T)\| = 0$. Therefore,

$$\|f(rT)\| \leqslant \left(C\frac{2}{1+r} + 1\right)\left\|\frac{f}{1-z}\right\|_{CSI(\mathbb{D})},$$

and, passing to the limit as $r \to 1$, we complete the proof. $\square$

Now, one can use the multipliers of the space $CSI(\mathbb{D})$. By definition,

$$Mult(\mathbb{D}) = \{f \in Hol(\mathbb{D}) :\ g \in CSI(\mathbb{D}) \Rightarrow fg \in CSI(\mathbb{D})\}.$$

The space $Mult(\mathbb{D})$ becomes a Banach algebra being equipped with the multiplier norm

$$||f||_{Mult} = \sup\{||fg||_{CSI(\mathbb{D})} : ||g||_{CSI(\mathbb{D})} \leqslant 1\}.$$

Basic facts on $Mult(\mathbb{D})$ can be found in Vinogradov et al. [VGH70]. In particular, $Hol(\overline{\mathbb{D}}) \subset Mult(\mathbb{D}) \subset H^\infty(\mathbb{D})$. The following corollary immediately follows from Theorem 1.2.

Corollary 1.3. *If $T \in (TR_C)$ then*

$$||f(T)|| \leqslant (C+1)||f||_{Mult(\mathbb{D})}$$

for every $f \in Hol(\overline{\mathbb{D}})$.

Remark 1.4. In fact, one can prove that given an operator $T \in (TR_C)$ and a function $f \in Hol(\overline{\mathbb{D}})$, we have

$$||f(T)|| \leqslant \left(\frac{C}{1-q} + 1\right)\left|\left|\frac{f}{1-z}\right|\right|_{CSI(\mathscr{B})}$$

for every $0 < q < 1$, where $\mathscr{B} = conv(0, D(0, \sin\arccos\frac{C}{q}))$. Using the classical Riesz turndown collar theorem (see [Lan46]), one can immediately infer that $\sup_{n\geqslant 0}||T^n|| < \infty$ and $\sup_{n\geqslant 0} n||T^n - T^{n+1}|| < \infty$. We refer to [Vitb] for details.

2. Polynomial free approximation and the norm of the polynomial calculus of a given degree n

Here we consider our main problem. Given a Banach space operator T, to determine the rate of the growth of the following constants $C_T(n)$:

$$C_T(n) = \sup\{||p(T)|| :\ \deg(p) \leqslant n, ||p||_\infty \leqslant 1\}.$$

First, we prove the following theorem.

Theorem 2.1. (1) *Let T be a Banach space operator satisfying the (TR) condition with a constant C. Then,*

$$C_T(n) \leqslant (C+1)\log(e^2 n)$$

for every $n \geqslant 1$.

(2) *There exists a Banach space operator T satisfying (TR) with a constant $C(T) \leqslant \frac{\pi}{2} + 1$ and such that*

$$C_T(n) \geqslant \frac{1}{2540} \log(e^2 n)$$

for every $n \geqslant 1$.

We start by proving part (1) of the theorem; it is a simple consequence of Section 1. Then we construct an example justifying assertion (2). As a part of the proof we obtain a result on the free interpolation by polynomials of a given degree; this will be stated and proved as an independent theorem (see Theorem 2.8 below). Later on, in Section 3, we give the analogues of assertion (2) for operators acting on some Banach spaces specified in advance. For some more properties of operators constructed in (2) see remarks and comments below in this section as well as in Section 3.

Proof of Theorem 2.1 (Assertion (1)). Let p be a polynomial of degree $n > 0$. We use the estimate of $||p(T)||$ in terms of the multiplier norm proved in Corollary 1.3: $||p(T)|| \leqslant (C+1)||p||_{Mult(\mathbb{D})}$. The map $f \mapsto pf$ on the space $CSI(\mathbb{D})$ is the adjoint operator of the map $g \mapsto P_+\bar{p}g$ on the space $\mathscr{C}_A(\mathbb{D})$ in the standard sesquilinear duality. Therefore, this map has the norm not exceeding $(2 + \log n)||p||_\infty$ since $||P_+\bar{p}|| \leqslant ||P_+||_{\bar{z}^n\mathscr{C}_A \to \mathscr{C}_A}|| \, ||p||_\infty$ and (we reproduce a well-known computation)

$$||P_+\bar{z}^n||_{\mathscr{C}_A \to \mathscr{C}_A} = ||z^n P_+ \bar{z}^n|| \leqslant ||I - P_{n-1}||_{L^\infty \to L^\infty},$$

where $P_{n-1}f = \sum_{k=0}^{n-1} \hat{f}(k) z^k$, and

$$\begin{aligned} ||P_{n-1}||_{L^\infty \to L^\infty} &= ||P_{n-1}||_{L^1 \to L^1} = \left|\left| \sum_{k=0}^{n-1} z^k \right|\right|_{L^1} = \frac{1}{\pi} \int_0^\pi \left| \sum_{k=0}^{n-1} e^{ikt} \right| dt \\ &\leqslant 1 + \frac{1}{\pi} \int_{\frac{\pi}{n}}^\pi \left| \frac{\sin(nt/2)}{\sin(t/2)} \right| dt \leqslant 1 + \int_{\frac{\pi}{n}}^\pi \frac{dt}{t} = 1 + \log n. \qquad \square \end{aligned}$$

Now we give an example showing that under the Tadmor–Ritt condition the growth rate of $C_T(n)$ obtained in assertion (1) cannot be improved over the class of all Banach space operators (assertion (2) of Theorem 2.1). We proceed by a series of lemmas and a theorem on free interpolation by polynomials of a given degree.

Let bv be the Banach space of complex sequences of bounded variation,

$$bv = \{(\lambda_k)_{k \geqslant 1} : Var((\lambda_k)_{k \geqslant 1}) < \infty\},$$

where $Var((\lambda_k)_{k \geqslant 1}) = \sum_{k \geqslant 1} |\lambda_{k+1} - \lambda_k|$. The norm on bv is given by

$$||(\lambda_k)_{k \geqslant 1}||_{bv} = Var((\lambda_k)_{k \geqslant 1}) + \left| \lim_{k \to \infty} \lambda_k \right|.$$

Now, let X be a Banach space with a basis $\mathscr{E} = (e_k)_{k\geqslant 1}$ and let T be an operator on X having e_k as eigenvectors, that is,

$$T = \sum\nolimits_{k\geqslant 1} \lambda_k(.,f_k)e_k, \tag{1}$$

where $(f_k)_{k\geqslant 1}$ is the biorthogonal sequence of $\mathscr{E}$, and $(\lambda_k)_{k\geqslant 1} \in bv$. In this case, as is well-known from McCarthy and Schwartz [MS65] or Markus [Mar66], we have

$$||T|| \leqslant B(\mathscr{E})||(\lambda_k)_{k\geqslant 1}||_{bv},$$

where $B(\mathscr{E})$ is the basis constant,

$$B(\mathscr{E}) = \sup_{n\geqslant 1} ||\mathscr{P}_{1,n}||.$$

Here $\mathscr{P}_{1,n}$ denotes the projection $\mathscr{P}_{1,n} = \sum_{k=1}^{n}(.,f_k)e_k$ onto $E_N = span\{e_k\}_{k=1}^{n}$. We also define the projections $\mathscr{P}_\sigma = \sum_{k\in\sigma}(.,f_k)e_k$, where $\sigma \subset \mathbb{N}^*$.

Lemma 2.2. *Let T be an operator defined by* (1). *If $(\lambda_k)_{k\geqslant 1} \subset (0,1)$ is an increasing sequence such that $\lim_{k\to\infty}\lambda_k = 1$, then T satisfies the Tadmor–Ritt condition with the constant $C = \left(\frac{\pi}{2}+1\right)B(\mathscr{E})$.*

Proof. For $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}$, the resolvent operator $R_\lambda(T)$ is of the same type,

$$R_\lambda(T) = \sum_{k\geqslant 1} \frac{1}{\lambda - \lambda_k}(.,f_k)e_k,$$

and

$$||R_\lambda(T)|| \leqslant B(\mathscr{E}) \left|\left| \left(\tfrac{1}{\lambda-\lambda_k}\right)_{k\geqslant 1} \right|\right|_{bv} = B(\mathscr{E})\left[Var\left(\left(\tfrac{1}{\lambda-\lambda_k}\right)_{k\geqslant 1}\right) + \lim_{k\to\infty} \frac{1}{|\lambda-\lambda_k|}\right].$$

Moreover, $\left(\frac{1}{\lambda-\lambda_n}\right)_{n\geqslant 1}$ is an ordered sequence varying on an arc γ of the circle $\left\{\frac{1}{\lambda-x} : x\in\mathbb{R}\right\}$. The endpoints of the arc are $\frac{1}{\lambda-\lambda_1}$ and $\frac{1}{\lambda-1}$. The arc γ is the smallest of the two arcs with these endpoints because the chord $\left[\frac{1}{\lambda-t}, \frac{1}{\lambda-1}\right]$, $0 \leqslant t < 1$, is always shorter than the diameter $\frac{1}{|Im\lambda|}$. Hence,

$$Var\left(\left(\frac{1}{\lambda-\lambda_k}\right)_{k\geqslant 1}\right) \leqslant |\gamma| \leqslant \frac{\pi}{2}\left|\frac{1}{\lambda-1} - \frac{1}{\lambda-\lambda_1}\right| = \frac{\pi}{2}\frac{1-\lambda_1}{|\lambda-\lambda_1||\lambda-1|} \leqslant \frac{\pi}{2|\lambda-1|}$$

for every λ, $|\lambda|>1$. $\square$

Lemma 2.3. *Let T be an operator defined by* (1) *on a Banach space X. If $(\lambda_k)_{k\geqslant 1}$ is a Carleson interpolating sequence and $(e_k)_{k\geqslant 1}$ is not an unconditional basis then T is not polynomially bounded over the unit disc $\mathbb{D}$, that is, $\sup_{n\geqslant 0} C_T(n) = \infty$.*

Proof. For $p \in \mathscr{P}ol$ we have $p(T) = \sum_{k\geqslant 1} p(\lambda_k)(.,f_k)e_k$. If we assume that T is polynomially bounded, $||p(T)|| \leqslant \beta ||p||_\infty$, where β is a constant not depending on p, then we can extend the polynomial calculus to H^∞ in the usual way, and we get $||f(T)|| \leqslant \beta ||f||_\infty$ for every $f \in H^\infty$, where $f(T)$ is defined by $f(T)e_k = f(\lambda_k)e_k$, $k \geqslant 1$. Take $(a_k)_{k\geqslant 1} \in l^\infty$. As $(\lambda_k)_{k\geqslant 1}$ is a Carleson sequence, there exists $f \in H^\infty$ such that $f(\lambda_k) = a_k$, $k \geqslant 1$. Thus the map $((x,f_k))_{k\geqslant 1} \mapsto (a_k(x,f_k))_{k\geqslant 1}$ is a bounded operator for every $(a_k)_{k\geqslant 1} \in l^\infty$, and therefore $(e_k)_{k\geqslant 1}$ is an unconditional basis. Contradiction. $\square$

Remark 2.4. The existence of a Hilbert space operator satisfying Tadmor–Ritt condition and not polynomially bounded was already proved by Le Merdy [LM98] using the same type of example. In fact, the above reasoning allows us to strengthen this example in the following way.

Corollary 2.5. *There exists a Hilbert space operator $T \in (TR)$* not *allowing a bounded polynomial calculus $||p(T)|| \leqslant const ||p||_{H^\infty(\Omega)}$ over any Jordan domain Ω* of the "*collar type*", *that is, such that $\mathbb{D} \subset \Omega$, $\mathbb{R}_+ = (0, \infty) \subset \mathbb{C} \backslash \overline{\Omega}$, see Fig.* 1.

Indeed, we can take an operator T of Lemma 2.3 constructed for a basis $(e_k)_{k\geqslant 1}$ which is not unconditional (see below about that). Since $\Lambda = (\lambda_k)_{k\geqslant 1}$ is an interpolating sequence, we have $\sup_{k\geqslant 1} \frac{1-\lambda_{k+1}}{1-\lambda_k} < 1$ by the Kabaila–Newman lemma, see details below. We claim that Λ is also an interpolating sequence for $H^\infty(Col)$, that is, $H^\infty(Col)|_\Lambda = l^\infty(\Lambda)$. Indeed, it suffices to show that $\varphi(\Lambda) = (\varphi(\lambda_k))_{k\geqslant 1}$ is interpolating for $H^\infty(\mathbb{D})$, where φ is the conformal mapping of Col onto $\mathbb{D}$. Without loss of generality, we may assume that $\partial\, Col$ is a piecewise smooth curve tangent

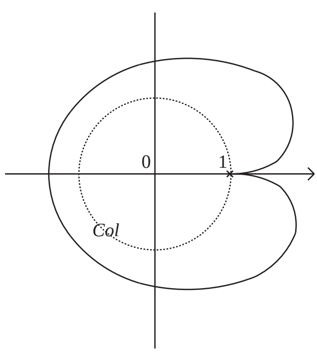

Fig. 1.

to $\mathbb{R}_+$ at point $z=1$. Then, we may assume that $\varphi(x)$ is real for every $x\in(0,1)$ and $\lim_{x\to 1}\varphi(x)=1$. Using the well-known asymptotics for conformal mappings (see, for example, [Pom75]) we get $1-\varphi(x)\sim(1-x)^{\frac{1}{2}}$ as $x\to 1$. Therefore, $\limsup_{k\to\infty}\frac{1-\varphi(\lambda_{k+1})}{1-\varphi(\lambda_k)}=\limsup_{k\to\infty}\left(\frac{1-\lambda_{k+1}}{1-\lambda_k}\right)^{\frac{1}{2}}<1$, and so $\varphi(\Lambda)$ is an interpolating sequence by the same Kabaila–Newman lemma.

Now, taking the same operator T as in Lemma 2.3 and assuming that there exists a constant $C>0$ such that $||p(T)||\leqslant C||p||_{H^\infty(Col)}$ for every polynomial p, we extend this calculus to $H^\infty(Col)$ (using that the polynomials are pointwise boundedly dense in $H^\infty(Col)$) and obtain a contradiction as in Lemma 2.3.

Now, we introduce the following *restricted unconditional basis constant* $UB(\mathscr{E}_N)$:

$$UB(\mathscr{E}_N)=\sup\{||\mathscr{P}_\sigma|_{E_N}||:\ \sigma\subset\{1,2,\dots,N\}\},$$

where $E_N=span\{e_k\}_{k=1}^N$.

Lemma 2.6. *Let T be an operator defined by* (1) *and let $n\geqslant 1$ be an integer. Assume that $N\geqslant 1$ is an integer and A a constant such that for every $\sigma\subset\{1,2,\dots,N\}$, there exists $p\in\mathscr{P}ol$,* $\deg(p)\leqslant n$, *such that*

$$p(\lambda_k)=\begin{cases}1 & if\ \ k\in\sigma\\ 0 & if\ \ k\in\{1,2,\dots,N\}\backslash\sigma\end{cases}\quad and\quad ||p||_\infty\leqslant A.$$

Then $UB(\mathscr{E}_N)\leqslant AC_T(n)$.

Proof. It is an immediate consequence of the fact that for such a p we have $p(T)|_{E_N}=\mathscr{P}_\sigma|_{E_N}$. $\square$

Now we recall a (known) example of a Banach space X having a basis $\mathscr{E}=(e_k)_{k\geqslant 1}$ such that $UB(\mathscr{E}_N)$ is of order N. Note that for any basis $\mathscr{E}$ we have $UB(\mathscr{E}_N)\leqslant B(\mathscr{E})N+1$ (use that $\min(card\ \sigma, card\ \sigma^c)\leqslant\frac{N}{2}$ for every $\sigma\subset\{1,\dots,N\}$).

Lemma 2.7. *Let $X=bv_0$ be the space of sequences of bounded variation tending to zero and $\mathscr{E}=(e_k)_{k\geqslant 1}$ the canonical* $0-1$ *basis in X. Then $B(\mathscr{E})=1$ and $UB(\mathscr{E}_N)\geqslant N$ for every $N\geqslant 1$.*

Proof. Let $a=(a_1,a_2,\dots)\in bv_0$. Then $\mathscr{P}_{(1,n)}a=(a_1,\dots,a_n,0,\dots)$ and

$$||\mathscr{P}_{(1,n)}a||_{bv_0}=\sum\nolimits_{k=1}^{n-1}|a_k-a_{k+1}|+|a_n|\leqslant\sum_{k=1}^{n-1}|a_k-a_{k+1}|+\sum_{k=n}^{m}|a_k-a_{k+1}|+|a_m|$$

for every $m\geqslant n$. Therefore $||P_{(1,n)}a||_{bv_0}\leqslant||a||_{bv_0}$, and hence $B(\mathscr{E})\leqslant 1$. Now let $\sigma=\sigma_{\text{odd}}=\{1,3,\dots,2N-1\}\subset\{1,2,\dots,2N-1\}$. Then $||\mathscr{P}_\sigma a||_{bv_0}=|a_1|+2(|a_3|+\cdots+|a_{2N-1}|)$. If $a=\sum_{k=1}^{2N-1}e_k$ then $||a||_{bv_0}=1$ and $||\mathscr{P}_\sigma a||_{bv_0}=2N-1$. Thus

$||\mathscr{P}_\sigma|_{E_{2N-1}}||\geqslant 2N-1$. Similarly, for $\sigma=\sigma_{\text{even}}=\{2,4,\dots,2N\}\subset\{1,2,\dots,2N\}$ and $a=\sum_{k=1}^{2N}e_k$ we get $||P_\sigma|_{E_{2N}}||\geqslant 2N$. Therefore $UB(\mathscr{E}_N)\geqslant N$ for every $N\geqslant 1$. $\square$

Next, we will prove that the conditions of Lemma 2.6 are satisfied for some N of order $log\, n$. For this we will take $\lambda_k=1-q^k$, $k\geqslant 1$, where $0<q<1$.

The rest of the reasoning has a more general nature and can represent an independent interest. Namely, we deal with the following polynomial free interpolation problem. Given a Carleson interpolating sequence $(\lambda_k)_{k\geqslant 1}$ and a (sufficiently large) number $A>0$, how fast can grow $N=N(n)$ such that every sequence $a=(a_k)_{k=1}^N$ can be interpolated by a polynomial p of degree less or equal to n, $p(\lambda_k)=a_k$, $k=1,\dots,N$, with the norm control $||p||_\infty\leqslant A||a||_\infty$? The answer is given by the following theorem. Recall that the Carleson constant of a sequence $\Lambda=(\lambda_k)_{k\geqslant 1}$ is, by definition,

$$\delta(\Lambda)=\inf_{k\geqslant 1}\prod_{j\neq k}|b_{\lambda_j}(\lambda_k)|,$$

where $b_\lambda(z)=\frac{\lambda-z}{1-\bar{\lambda}z}$. A sequence Λ is interpolating, i.e. $H^\infty|_\Lambda=l^\infty$, if and only if $\delta(\Lambda)>0$, see [Gar81] or [Nik02, Vol. 2].

Theorem 2.8. *Let $\Lambda=(\lambda_k)_{k\geqslant 1}$ be an interpolating sequence having the Carleson constant $\delta=\delta(\Lambda)>0$. Then for every r, $\frac{1}{2}<r<1$, and every sequence $a=(a_k)_{k\geqslant 1}\in l^\infty$ there exists a polynomial p, $\deg(p)\leqslant\frac{M}{1-r}$, where $M=\frac{9\pi}{4\delta^{32}}$, such that*

(a) *$p(\lambda_k)=a_k$ for all λ_k with $|\lambda_k|\leqslant r$,*
(b) *$||p||_\infty\leqslant\frac{80}{\delta^{32}}||a||_\infty$.*

Note that the upper bound $\deg(p)\leqslant\frac{const}{1-r}$ for the degree of free polynomial interpolation on $\Lambda_r=\{\lambda_k:|\lambda_k|\leqslant r\}$ is sharp as $r\to 1$ in the sense of Corollary 2.12. Moreover, using Remark 2.10 below, one can improve asymptotically, as $r\to 1$, the value of the constants. Namely, for every $\varepsilon>0$ there exists $0<R<1$ such that (in the notation of Theorem 2.8) $M\leqslant\frac{9\pi}{4\delta^{8+\varepsilon}}$ and $||p||_\infty\leqslant\frac{80}{\delta^{8+\varepsilon}}||a||_\infty$ for every r, $R<r<1$.

As to the growth of $card(\Lambda_r)$ as $r\to 1$, it can vary considerably. For instance, for the most dense interpolating sequences on the interval $(0,1)$, it should be $card(\Lambda_r)=O\left(\log\frac{1}{1-r}\right)$, see below the proof of assertion (2) of Theorem 2.1. In general, as is well-known, for every $r_k>0$ such that $\Sigma=\sum_{k\geqslant 1}(1-r_k)<\infty$, there exists an interpolating sequence $\Lambda=(\lambda_k)_{k\geqslant 1}$ with $r_k=|\lambda_k|$ for every $k\geqslant 1$ (Naftalevich's theorem, see [Nik86]). Therefore, the best general estimate is given by the Tchebyshev inequality $card\,\Lambda_r\leqslant\frac{\Sigma}{1-r}$. In fact, $card\,\Lambda_r=o\left(\frac{1}{1-r}\right)$ as $r\to 1$.

Yet another comment on Theorem 2.8 is that given a finite sequence $\Lambda=(\lambda_j)_{j=1}^N$, we can insure the interpolation in the sense (a) and (b) on the entire Λ by polynomials of degree $d \leqslant \frac{M}{1-r}$, where $r = \max_{1 \leqslant j \leqslant N} |\lambda_j|$ and $M = \frac{9\pi}{4\delta(\Lambda)^{32}}$. This bound for the degree d does not seem to be sharp. Besides, we can compare Theorem 2.8 with a result of Bourgain [Bou86] saying that given points $\lambda_1, \ldots, \lambda_m$ in $\{z : |z| < r\}$ and a function $f \in H^\infty$, there exists a polynomial p with $\deg(p) \leqslant d = \frac{m^4}{(1-r)^2}$, $p(\lambda_j) = f(\lambda_j)$, $1 \leqslant j \leqslant m$, and $||p||_\infty \leqslant 3||f||_\infty$. Adapting the result to an interpolating sequence, we get a free interpolation on $(\lambda_j)_{j=1}^m$ by polynomials of degree $d \leqslant \frac{const}{(1-r)^6}$.

The proof of Theorem 2.8 depends on two lemmas.

Lemma 2.9. *Let $\Lambda = (\lambda_k)_{k \geqslant 1}$ be an interpolating sequence and let $r\Lambda_r = \{r\lambda_k : |\lambda_k| \leqslant r\}$, where $0 < r < 1$. Then, $r\Lambda_r$ are interpolating sequences and*

$$\delta(r\Lambda_r) \geqslant \delta(\Lambda)^{16}$$

for $\frac{1}{2} \leqslant r < 1$.

Proof. Let $|\lambda_k| \leqslant r$, $\frac{1}{2} \leqslant r < 1$. Since $\frac{1}{|b_\lambda(\mu)|^2} = 1 + \frac{(1-|\lambda|^2)(1-|\mu|^2)}{|\lambda-\mu|^2}$ for $\lambda \neq \mu$, we have

$$\begin{aligned}
\prod_{\lambda_j \in \Lambda_r \setminus \{\lambda_k\}} \frac{1}{|b_{r\lambda_j}(r\lambda_k)|^2} &= \prod_{\lambda_j \in \Lambda_r \setminus \{\lambda_k\}} \left(1 + \frac{(1 - r^2|\lambda_j|^2)(1 - r^2|\lambda_k|^2)}{r^2|\lambda_j - \lambda_k|^2}\right) \\
&\leqslant \prod_{\lambda_j \in \Lambda_r \setminus \{\lambda_k\}} \left(1 + 4\frac{(1 - |\lambda_j|^4)(1 - |\lambda_k|^4)}{|\lambda_j - \lambda_k|^2}\right) \\
&\leqslant \prod_{\lambda_j \in \Lambda_r \setminus \{\lambda_k\}} \left(1 + 16\frac{(1 - |\lambda_j|^2)(1 - |\lambda_k|^2)}{|\lambda_j - \lambda_k|^2}\right) \\
&\leqslant \prod_{\lambda_j \in \Lambda_r \setminus \{\lambda_k\}} \left(1 + \frac{(1 - |\lambda_j|^2)(1 - |\lambda_k|^2)}{|\lambda_j - \lambda_k|^2}\right)^{16} \\
&= \left(\prod_{\lambda_j \in \Lambda_r \setminus \{\lambda_k\}} \frac{1}{|b_{\lambda_j}(\lambda_k)|^2}\right)^{16} \leqslant \frac{1}{\delta^{32}},
\end{aligned}$$

and the result follows. □

Remark 2.10. It is worth mentioning that the same reasoning but using

$$\frac{(1 - r^2|\lambda_j|^2)(1 - r^2|\lambda_k|^2)}{r^2|\lambda_j - \lambda_k|^2} \leqslant \frac{4}{r^2}\frac{(1 - |\lambda_j|^2)(1 - |\lambda_k|^2)}{|\lambda_j - \lambda_k|^2}$$

shows that asymptotically, as $r \to 1$, we have $\liminf_{r \to 1} \delta(r\Lambda_r) \geqslant \delta(\Lambda)^4$.

Lemma 2.11. *Let $\Lambda = (\lambda_k)_{k=1}^N \subset \mathbb{D}$ be a sequence of distinct points, $0 < r < 1$ and $\delta_r = \delta(r\Lambda)$. Then, for every sequence $(a_k)_{k=1}^N$, there exists a polynomial p such that*

(1) $\deg(p) \leqslant \frac{M}{1-r}$, *where* $M = \frac{9\pi}{2\delta_r^2}$,
(2) $p(\lambda_k) = a_k$, $1 \leqslant k \leqslant N$,
(3) $||p||_\infty \leqslant \frac{80}{\delta_r^2} ||a||_\infty$.

Proof. Let $(a_k)_{k=1}^N$, $|a_k| \leqslant 1$. As $(r\lambda_k)_{k=1}^N$ is a Carleson set with the constant $\delta_r > 0$ there exists $h \in H^\infty$ such that $h(r\lambda_k) = a_k$, $1 \leqslant k \leqslant N$, and $||h||_\infty \leqslant A_r$, where $A_r = \frac{8}{\delta_r^2}$ (see [Gar81] or [Nik02, Vol. 2]). Let $h_r(z) = h(rz)$. Then

$$\begin{cases} h_r(\lambda_k) = a_k, & 1 \leqslant k \leqslant N, \\ ||h_r||_\infty \leqslant A_r. \end{cases}$$

Now, we approximate h_r by polynomials using a version of the Jackson theorem due to Favard, Akhiezer and Krein (see [Tim94, Sections 5.5.1–5.5.4]). Namely, the theorem says that there exists a trigonometric polynomial K_n on $\mathbb{T}$ such that $\deg(K_n) \leqslant n$ and $||f - K_n * f||_\infty \leqslant \frac{\pi}{2}\frac{1}{n+1} ||f'||_\infty$ for every function f on $\mathbb{T}$ having $f' \in L^\infty(\mathbb{T})$. Applying this to $f = h_r$ and $n = \left[\frac{M}{1-r}\right]$ we get an analytic polynomial $p = K_n * h_r$ with $\deg(p) \leqslant n$ and such that

$$||h_r - p||_\infty \leqslant \frac{\pi}{2}\frac{1}{n+1} ||h_r'||_\infty.$$

Since

$$|h_r'(z)| = r|h'(rz)| \leqslant r\frac{||h||_\infty}{1-|rz|^2} \leqslant \frac{r||h||_\infty}{1-r^2} \leqslant \frac{A_r}{2(1-r)},$$

we obtain

$$||h_r - p||_\infty \leqslant \frac{\pi}{4}\frac{1}{n+1}\frac{A_r}{1-r} < \frac{\pi}{4}\frac{A_r}{M} = \frac{\pi}{4}\frac{8}{M\delta_r^2} = \frac{8}{9}.$$

Setting $\varepsilon = \frac{8}{9}$, we have $||h_r - p||_\infty \leqslant \varepsilon$, thus $||p||_\infty \leqslant \varepsilon + ||h_r||_\infty \leqslant \varepsilon + A_r$.

As above, we denote $l^\infty(\Lambda)$ the space of bounded functions on Λ.

We have proved the following approximation property. *For every $a \in l^\infty(\Lambda)$ there exists $p \in \mathscr{P}ol$, $\deg(p) \leqslant n$, such that*

$$\begin{cases} ||a - p|_\Lambda||_\infty \leqslant \varepsilon ||a||_\infty, \\ ||p||_\infty \leqslant (\varepsilon + A_r)||a||_\infty. \end{cases}$$

Now, we obtain the interpolating polynomial by successive approximations. Let $a \in l^\infty(\Lambda)$, and let $p_1 \in \mathscr{P}ol$, $\deg(p_1) \leqslant n$, such that

$$\begin{cases} \|a - p_1|_\Lambda\|_\infty \leqslant \varepsilon \|a\|_\infty, \\ \|p_1\|_\infty \leqslant (\varepsilon + A_r)\|a\|_\infty. \end{cases}$$

We apply the above approximation to $a - p_1|_\Lambda$. Let $p_2 \in \mathscr{P}ol$, $\deg(p_2) \leqslant n$, such that

$$\begin{cases} \|a - p_1|_\Lambda - p_2|_\Lambda\|_\infty \leqslant \varepsilon \|a - p_1|_\Lambda\|_\infty \leqslant \varepsilon^2 \|a\|_\infty, \\ \|p_2\|_\infty \leqslant (\varepsilon + A_r)\|a - p_1|_\Lambda\|_\infty \leqslant (\varepsilon + A_r)\varepsilon\|a\|_\infty. \end{cases}$$

By induction, there exists $(p_k)_{k \geqslant 1} \subset \mathscr{P}ol$ such that

$$\begin{cases} \deg(p_k) \leqslant n, \;\; k \geqslant 1, \\ \|a - \sum_{j=1}^k p_j|_\Lambda\|_\infty \leqslant \varepsilon^k \|a\|_\infty, \;\; k \geqslant 1, \\ \|p_k\|_\infty \leqslant (\varepsilon + A_r)\varepsilon^{k-1}\|a\|_\infty, \;\; k \geqslant 1. \end{cases}$$

In these conditions, the series $\sum_{k \geqslant 1} p_k$ converges and $p = \sum_{k \geqslant 1} p_k$ is a polynomial of degree less or equal to n such that

$$\begin{cases} p(\lambda_k) = a_k, \;\; 1 \leqslant k \leqslant N, \\ \|p\|_\infty \leqslant \dfrac{\varepsilon + A_r}{1 - \varepsilon}\|a\|_\infty \leqslant 9\left(\dfrac{8}{9} + \dfrac{8}{\delta_r^2}\right) \leqslant \dfrac{80}{\delta_r^2}. \end{cases} \qquad \square$$

Proof of Theorem 2.8. By Lemma 2.9, $\delta_r = \delta(r\Lambda_r) \geqslant \delta(\Lambda)^{16}$ for every r, $\frac{1}{2} \leqslant r < 1$. Applying Lemma 2.11 to $\Lambda_r = \{\lambda_k : |\lambda_k| \leqslant r\}$ we get a polynomial p of degree $\deg(p) \leqslant \frac{M_r}{1-r} \leqslant \frac{M}{1-r}$, where $M = \frac{9\pi}{4\delta(\Lambda)^{32}}$, such that $p(\lambda_k) = a_k$ for every λ_k, $|\lambda_k| \leqslant r$, and $\|p\|_\infty \leqslant \frac{80}{\delta_r^2} \leqslant \frac{80}{\delta(\Lambda)^{32}}$. $\square$

Now we return to the proof of Theorem 2.1. In order to complete the construction of an example for part (2) of the theorem, we need a particular case of Theorem 2.8 on polynomial interpolation, namely, the case when $0 < \lambda_k < 1$, $\lambda_{k+1} > \lambda_k$. By the way, in this case, it is known that $(\lambda_k)_{k \geqslant 1}$ is a Carleson interpolating sequence if and only if $q = \sup_{k \geqslant 1} \frac{1-\lambda_{k+1}}{1-\lambda_k} < 1$ (Kabaila, Newman, see [Nik86]). To make the example more explicit we simply take $\lambda_k = 1 - q^k$, $k \geqslant 1$, where $0 < q < 1$. In this case, $\delta(\Lambda) \geqslant \prod_{m \geqslant 1} \left(\frac{1-q^m}{1+q^m}\right)^2$ (see the proof of the Kabaila–Newman lemma in [Nik86]). It also can be proved that

$$\sup\left\{\frac{1 - r\lambda_{k+1}}{1 - r\lambda_k} : \lambda_{k+1} \leqslant r\right\} \leqslant q_0 = \frac{2+q}{3},$$

and hence $\delta_r = \delta(r\Lambda_r) \geqslant \prod_{m\geqslant 1}\left(\frac{1-q_0^m}{1+q_0^m}\right)^2$ for every r, $\frac{1}{2}\leqslant r<1$. This estimate looks better than that of Lemma 2.9 above, $\delta_r \geqslant \delta(\Lambda)^{16}$, but we will not use it for the proof of assertion (2) of Theorem 2.1 for optimization reasons (we need quite a small value of q_0 whereas in the above case we would always have $q_0 \geqslant \frac{2}{3}$).

Proof of assertion (2) of Theorem 2.1. Let $0<q<1$ and $\lambda_k = 1-q^k$, $k=1,2,\dots$. Let $\frac{1}{2}\leqslant r<1$ and $n=\left[\frac{M}{1-r}\right]$, where $M=\frac{9\pi}{4\delta(\Lambda)^{32}}$ (we use the notation of the previous lemmas). Then n is an arbitrary integer with $n\geqslant [2M]$. Using Lemma 2.11 we take a polynomial p with $\deg(p)\leqslant n$, $||p||_\infty \leqslant \frac{80}{\delta(\Lambda)^{32}}$ and $p(\lambda_{2k})=1$, $p(\lambda_{2k+1})=0$ for all k such that $\lambda_{2k+1}\leqslant r$. For our case the number of λ_j with $\lambda_j = 1-q^j\leqslant r$ is

$$N(r) = \left[\frac{\log\frac{1}{1-r}}{\log\frac{1}{q}}\right] \geqslant \frac{\log\frac{n}{M}}{\log\frac{1}{q}} - 1 \geqslant \frac{1}{2}\frac{\log n}{\log\frac{1}{q}},$$

at least if $\log n \geqslant 2\log M + \log\frac{1}{q}$, i.e. if $n\geqslant \frac{M^2}{q} = \frac{(9\pi)^2}{4q\delta(\Lambda)^{64}} = n(q)$.

Let T be the same operator on the space bv_0 as in Lemma 2.7. As in the proofs of Lemmas 2.6 and 2.7 we have

$$||p(T)|| \geqslant N(r) \geqslant \frac{1}{2}\frac{\log n}{\log\frac{1}{q}},$$

and hence

$$C_T(n) \geqslant \frac{\delta(\Lambda)^{32}}{160}\frac{\log n}{\log\frac{1}{q}}$$

for $n\geqslant n(q)$. Next, we use the following estimate for $\delta(\Lambda)$:

$$\begin{aligned}\log\frac{1}{\delta(\Lambda)} &\leqslant 2\sum_{m\geqslant 1}\log\left(1+\frac{2q^m}{1-q^m}\right) \leqslant \sum_{m\geqslant 1}\frac{4q^m}{1-q^m}\\ &\leqslant \frac{4}{1-q}\sum_{m\geqslant 1} q^m = \frac{4q}{(1-q)^2}.\end{aligned}$$

This implies

$$C_T(n) \geqslant e^{-2^7 q(1-q)^{-2}}\frac{\log n}{160\log\frac{1}{q}}$$

for $n\geqslant n(q)$. Taking $q=2^{-7}$ we get

$$C_T(n) \geqslant e^{\frac{-1}{(1-2^{-7})^2}}\frac{\log n}{160\cdot 7\log 2} \geqslant \frac{0,36211}{160\cdot 7\log 2}\log n \geqslant \frac{1}{2150}\log n \geqslant \frac{1}{2540}\log(e^2 n)$$

for $n \geqslant n(q) = \frac{(9\pi)^2}{4q\delta(\Lambda)^{64}}$. Since $\frac{(9\pi)^2}{4q\delta(\Lambda)^{64}} \leqslant \frac{(9\pi)^2}{4q} e^{\frac{2^8 q}{(1-q)^2}} \leqslant \frac{(9\pi)^2}{4} \cdot 2^7 \cdot 2^{32} = 2^{39} \cdot 3^4 \cdot \pi^2 = n_0$, the above lower estimate is valid for $n \geqslant n_0$. It is also easy to see that $\frac{1}{2540} \log(e^2 n) \leqslant 1 \leqslant C_T(n)$ for $n \leqslant n_0$. $\square$

As is already mentioned, the above proof heavily depends on Theorem 2.8 for polynomial free interpolation. It is curious to note that, in turn, points (1) and (2) of Theorem 2.1 imply that the logarithmic growth rate of a subset $\Lambda_n \subset (0,1)$ where the free interpolation by polynomials of degree $\leqslant n$ is possible, is sharp. More precisely, we have the following corollary.

Corollary 2.12. *Let $(\lambda_k)_{k \geqslant 1}$ be a (finite or infinite) sequence in the interval $(0,1)$, $0 < \lambda_k < \lambda_{k+1} < 1$, let $m(n)$, $n \geqslant 1$, be an integer and $c > 0$ a constant such that for every $a = (a_k)_{k \geqslant 1} \in l^\infty$ there is a polynomial p satisfying $\deg(p) \leqslant n$ and*

(a) $p(\lambda_k) = a_k,\ k = 1, \dots, m(n)$,
(b) $\|p\|_\infty \leqslant c\|a\|_\infty$.

Then, $m(n) \leqslant 4c \log(e^2 n)$ for $n \geqslant 1$.

Indeed, it is clear from Lemmas 2.6 and 2.7 and the proof of assertion (2) of Theorem 2.1 that, under the conditions of the corollary, there exists a Banach space operator T satisfying the (TR) condition and such that $m(n) \leqslant C_T(n)c$. Using assertion (1) of the same theorem we have $C_T(n) \leqslant \left(\frac{\pi}{2} + 2\right) \log(e^2 n)$. The result follows.

Remark 2.13. It is worth mentioning some additional properties of operators constructed in point (2) of Theorem 2.1, as well as to compare the constants $C_T(n)$ with similar bounds for *rational calculus*.

(1) Let T be an operator constructed in Theorem 2.1(2). By formula (1) at the beginning of Section 2,

$$\begin{aligned} \|T - I\|_{S_1} &= \left\| \sum\nolimits_{k \geqslant 1} (\lambda_k - 1)(\cdot, f_k) e_k \right\|_{S_1} \leqslant \sum\nolimits_{k \geqslant 1} |\lambda_k - 1| \, \|f_k\| \, \|e_k\| \\ &\leqslant 2B(\mathscr{E}) \sum\nolimits_{k \geqslant 1} |\lambda_k - 1|. \end{aligned}$$

Therefore, $T - I$ is a trace class operator with an arbitrary small trace norm (we can start taking eigenvalues from λ_N for an arbitrary N, instead of λ_1). On the other hand, it is easy to see that every Ritt operator T with $rank(T - I) < \infty$ is polynomially bounded ($\|p(T)\| \leqslant const \|p\|_\infty$ for every polynomial p).

(2) For such an operator T it is easy to see that $\sigma(T) = \{1\} \cup \{\lambda_k : k \geqslant 1\}$ and

$$\|R_z(T)\| \leqslant \frac{const}{dist(z, \sigma(T))}$$

for every $z \in \mathbb{C} \backslash \sigma(T)$.

(3) It is of interest to compare the calculi possessed by Ritt and Kreiss operators. As is clear from comments in the Introduction, these classes are quite different with respect to the polynomial calculus: we have always $C_T(n) \leqslant const \log(en)$ for Ritt operators, but only $C_T(n) \leqslant const\ n$ for Kreiss ones (both are sharp estimates). If we define similar constants for the *rational* calculus,

$$RC_T(n) = \sup\{||r(T)|| : ||r||_\infty \leqslant 1, r \in C_A(\mathbb{D}), r \text{ is a rational function, } \deg(r) \leqslant n\},$$

then we still have $RC_T(n) \leqslant const\ n$ for every $n \geqslant 1$ and for every Kreiss operator T (see [Vita]). Now, we can see that the same bound is sharp even for Ritt operators. Namely, let T be an operator from Theorem 2.1(2) and $f \in H^\infty$ such that $f(\lambda_k) = 1$ for k odd, $f(\lambda_k) = 0$ for k even. By the Nevanlinna–Pick theorem, for every $n \geqslant 1$, there exists a rational function r_n such that $r_n(\lambda_k) = f(\lambda_k)$, $1 \leqslant k \leqslant n$, $\deg(r_n) \leqslant n$ and $||r_n||_\infty \leqslant ||f||_\infty$. Therefore, as in Theorem 2.1(2), we get $||r_n(T)|| \geqslant const\ UB(\mathscr{E}_n) \geqslant const\ n$, and hence $RC_T(n) \geqslant const\ n$ for every $n \geqslant 1$.

For operators defined on specific Banach spaces (including Hilbert spaces) we refer to Section 3 below; for instance, for every $\varepsilon > 0$, there exists a Hilbert space Ritt operator such that $RC_T(n) \geqslant c_\varepsilon n^{1-\varepsilon}$, $n \geqslant 1$.

We finish this section with an application of the preceding results to a question of complex analysis. Recall first that the behavior of the integrals (the radial variation of f)

$$V(f, \zeta) = \int_0^1 |f'(r\zeta)|\, dr, \quad \zeta \in \mathbb{T},$$

is quite important for the geometric function theory, as well as for the comparison of some function classes in the unit disc $\mathbb{D}$, see, for instance, [Nik02, Vol.1, p. 376]. In particular, Rudin [Rud55] has exhibited a function $f \in \mathscr{C}_A(\mathbb{D})$ such that $V(f, \zeta) = \infty$ for a.e. $\zeta \in \mathbb{T}$. In this framework, it is of interest to know how $V(f, \zeta)$ behaves for "good functions", in particular for polynomials of a given degree. Let

$$V_n = \sup_{\substack{\deg(p) \leqslant n \\ ||p||_\infty \leqslant 1}} \left\{ |p(1)| + \int_0^1 |p'(t)|\, dt \right\}.$$

Corollary 2.14. *For every* $n \geqslant 1$,

$$\frac{1}{2540} \log(e^2 n) \leqslant V_n \leqslant 4 \log(e^2 n).$$

Indeed, notice first that the space bv_0 introduced in Lemma 2.7 is an algebra according to the usual multiplication $(x_k)_{k \geqslant 1}(y_k)_{k \geqslant 1} = (x_k y_k)_{k \geqslant 1}$. In fact, we have $||xy||_{bv_0} \leqslant 2||x||_{bv_0}||y||_{bv_0}$ for every $x, y \in bv_0$. Moreover, if M_y denotes an operator of multiplication by an element $y \in bv$, then $||y||_{bv} \leqslant ||M_y|| \leqslant 2||y||_{bv}$.

We define T on bv_0 by $T(x_k)_{k\geqslant 1} = (\lambda_k x_k)_{k\geqslant 1}$, where $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N = \lambda_{N+1} = \cdots = 1$. From Lemmas 2.2 and 2.7 we know that T satisfies the (TR) condition with the constant $C = \frac{\pi}{2} + 1$. Then $p(T)$ is an operator of the multiplication by $(p(\lambda_k))_{k\geqslant 1}$, $p(T)(x_k)_{k\geqslant 1} = (p(\lambda_k)x_k)_{k\geqslant 1}$, and by the above remark

$$||(p(\lambda_k))||_{bv} = \sum_{k\geqslant 1} |p(\lambda_{k+1}) - p(\lambda_k)| + |p(1)| \leqslant ||p(T)|| \leqslant 2||(p(\lambda_k))||_{bv}.$$

Theorem 2.1(1) implies that $||p(T)|| \leqslant (C+1)\log(e^2 n)||p||_\infty$ if $\deg(p) \leqslant n$. Thus, $|p(1)| + \sum_{k\geqslant 1} |p(\lambda_{k+1}) - p(\lambda_k)| \leqslant (C+1)\log(e^2 n)||p||_\infty$. Now, we take the limit over all subdivisions $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N = 1$, $N \to \infty$, and get $|p(1)| + \int_0^1 |p'(t)|\, dt \leqslant 4\log(e^2 n)||p||_\infty$ for every polynomial p with $\deg(p) \leqslant n$.

On the other hand, for every T defined as above we have

$$||p(T)|| \leqslant 2\Big(|p(1)| + \textstyle\sum_{k\geqslant 1} |p(\lambda_k) - p(\lambda_{k+1})|\Big) \leqslant 2\left(|p(1)| + \displaystyle\int_0^1 |p'(t)|\, dt\right)$$

for every polynomial p. From the proof of Theorem 2.1(2) we know that

$$\sup||p(T)|| \geqslant \frac{1}{2540}\log(e^2 n),$$

where the sup is taken over all polynomials p with $\deg(p) \leqslant n$, $||p||_\infty \leqslant 1$, and all T corresponding to all subdivisions $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N = 1$. The corollary follows.

Remark 2.15. By the way, Corollary 2.14 suggests yet another example of an operator T satisfying (TR) and such that $C_T(n) \geqslant a\log n$, $n \geqslant 1$. Indeed, let W_1^1 be the space of absolutely continuous functions on the interval $(0,1)$ having $f' \in L^1(0,1)$ and endowed with the norm $||f|| = |f(0)| + \int_0^1 |f'(x)|dx$. Let

$$Tf(x) = xf(x), \quad f \in W_1^1.$$

Since W_1^1 is an algebra with respect to the pointwise multiplication and $||fg|| \leqslant 2||f||\,||g||$ for every $f, g \in W_1^1$, we get $||R_z(T)|| \leqslant 2\left|\left|\frac{1}{z-x}\right|\right| \leqslant \frac{b}{|z-1|}$ for an absolute constant $b > 0$ and for every z, $|z| > 1$. Thus, $T \in (TR)$.

On the other hand, $||p(T)|| \geqslant ||p||_{W_1^1}$ and hence, by Corollary 2.14, we have $C_T(n) \geqslant V_n \geqslant a\log n$ for every $n \geqslant 1$.

3. Hilbert spaces and other specific Banach spaces

Here we are looking for "worst" Tadmor–Ritt operators acting on a given Banach space X. To this end, besides the quantities $C_T(n)$ used in Section 2, we consider a characteristic of the space X defined by

$$C(X,n) = \sup\left\{\frac{C_T(n)}{C(T)} : T : X \to X,\ T \in (TR)\right\}$$

for $n = 1, 2, \ldots$. For possible applications, the most interesting cases are $X = L^p(\mu)$, $1 \leqslant p \leqslant \infty$, including Hilbert spaces. For these cases we obtain a partial information, see Lemma 3.3 and Theorem 3.4 below. The techniques used are still the same as in Section 2, that is, basis and unconditional basis constants, $B(\mathscr{E})$ and $UB(\mathscr{E}_N)$ respectively.

Before starting we mention two obvious properties: $B(\mathscr{E}) = B(\mathscr{E}^*)$ and $UB(\mathscr{E}_N) = UB(\mathscr{E}_N^*)$, where $\mathscr{E} = (e_k)_{k\geqslant 1}$ is a basis in a Banach space X and $\mathscr{E}^* = (e_k^*)_{k\geqslant 1}$ is the dual basis in X^* (maybe, in a weak sense). Recall also that always $UB(\mathscr{E}_N) \leqslant NB(\mathscr{E}) + 1$.

First, we need to construct finite bases $\mathscr{E} = \mathscr{E}^N$ whose unconditional basis constant grows linearly in both N and $B(\mathscr{E}^N)$.

Lemma 3.1. *Let* $b \geqslant 1$ and $l^1_{N,b}$ *be the space* $\mathbb{C}^N$ *endowed with the norm*

$$||(x_k)_{k=1}^N||_b = |x_1| + b\sum_{k=1}^{N-1} |x_k - x_{k+1}|.$$

Let $\mathscr{E}^N = (e_k)_{k=1}^N$ *be the standard* $0-1$ *basis in* $\mathbb{C}^N$. *Then*

(1) *the space* $l^1_{N,b}$ *is isometrically isomorphic to* $l^1_N = (\mathbb{C}^N, ||\cdot||)$, $||x|| = \sum_{k=1}^N |x_k|$, $x \in \mathbb{C}^N$;

(2) $B(\mathscr{E}^N) \leqslant 1 + b$, *and* $B(\mathscr{E}^N) = 1 + b$ *for* $N \geqslant 3$, *and*

$$\frac{1}{2} N \cdot B(\mathscr{E}^N) \leqslant 1 + b(N-1) \leqslant UB(\mathscr{E}^N) \leqslant N(b+1) + 1 = N \cdot B(\mathscr{E}^N) + 1;$$

(3) *by duality, the same quality basis as in* (2) *exists in the space* l^∞_N.

Proof. (1) Clearly, the mapping $Vx = (x_1, b(x_1 - x_2), \ldots, b(x_{n-1} - x_n))$ is an isometry from $l^1_{N,b}$ onto l^1_N.

(2) Since $x_n = x_1 + \sum_{k=1}^{n-1} (x_{k+1} - x_k)$ for every n, $1 < n < N$, we get

$$||P_{(1,n)}x||_b = |x_1| + b\sum_{k=1}^{n-1} |x_k - x_{k+1}| + b|x_n| \leqslant (1+b)|x_1| + 2b\sum_{k=1}^{n-1} |x_k - x_{k+1}| \leqslant (1+b)||x||_b$$

for every $x \in l^1_{N,b}$. Hence, $B(\mathscr{E}^N) \leqslant 1 + b$.

On the other hand, for $x_0 = (1, 1, \dots, 1)$, we have $||x||_b = 1$, and $||P_{(1,n)}x_0|| = 1 + b$ for $1 < n < N$.

Next, for $\sigma = \sigma_{\text{odd}}$ the odd part of $\{1, 2, \dots, N\}$, we get $||P_\sigma x_0|| = 1 + b(N-1)$ and hence $UB(\mathscr{E}^N) \geqslant 1 + b(N-1) \geqslant \frac{1}{2}N(b+1)$ because $b(N-2) + 2 \geqslant N$ for $b \geqslant 1$ and $N \geqslant 2$.

On the other hand, $UB(\mathscr{E}^N) \leqslant NB(\mathscr{E}) + 1 \leqslant N(1+b) + 1$. $\square$

For Hilbert spaces, we have only the following lemma which is mostly known; property (1) is proved in McCarthy and Schwartz [MS65] and property (2) in Spijker et al. [STW03].

Lemma 3.2. *Let H be a Hilbert space.*

(1) *If $\mathscr{E} = (e_k)_{k \geqslant 1}$ is a basic sequence in H (i.e. a basis in span $\mathscr{E}$) and $\mathscr{E}_N = (e_k)_{k=1}^N$ then $UB(\mathscr{E}_N) \leqslant c_1 B(\mathscr{E}_N) N^{1-b}$, where $b \geqslant \frac{c_2}{B(\mathscr{E}_N)^2}$ and c_1, $c_2 > 0$ are absolute constants.*
(2) *Given $B > 1$, there exists a basis $\mathscr{E} = (e_k)_{k \geqslant 1}$ in H such that $B(\mathscr{E}) \leqslant B$ and $UB(\mathscr{E}^N) \geqslant \frac{N^{1-\frac{1}{B}}}{6\sqrt{B}}$ for every $N \geqslant 1$, where $\mathscr{E}_N = (e_k)_{k=1}^N$. In fact, one can take $H = L^2(\mathbb{T}, wdm)$ and $e_k = e^{ikt}$, where w is a suitable weight on the unit circle $\mathbb{T}$.*

Proof. (1) It is proved in [MS65] that $UB(\mathscr{E}^N) \leqslant B(2N)^{1-\log_2\left(1+\frac{1}{B^2}\right)}$, where $B = \sup\{||I - 2P_{(m,n)}|| : m \leqslant n\}$. Since $B \leqslant 1 + 2\sup ||P_{(m,n)}|| \leqslant 1 + 4B(\mathscr{E}^N) \leqslant 5B(\mathscr{E}^N)$, the result follows.

For (2) we refer to [STW03]. $\square$

Lemma 3.3. (1) *Let X be a Banach space containing a complemented subspace isomorphic to l^1 or c_0. There exist constants $\alpha, \beta, \gamma, \delta > 0$ such that for every $B > 1$ there is an operator $T : X \to X$ satisfying (TR) with $\alpha B \leqslant C(T) \leqslant \beta B$ and*

$$\gamma C(T) \log(en) \leqslant C_T(n) \leqslant \delta C(T) \log(en)$$

for every $n \geqslant 1$.

In particular, this is the case for every infinite dimensional space $L^1(\mu)$, $L^\infty(\mu)$, $\mathscr{C}(K)$.

(2) *Let X be a Banach space containing an infinite dimensional complemented subspace isomorphic to a Hilbert space. There exist constants $\alpha, \beta > 0$ depending only on X such that for every $B > 1$ there is an operator $T : X \to X$ satisfying (TR) with $C(T) \leqslant \alpha B$ and $C_T(n) \geqslant \beta B^{-\frac{1}{2}} (\log(en))^{1-\frac{1}{B}}$.*

In particular, this is the case for spaces $L^p(\mu)$, $1 < p < \infty$, with any non-purely atomic measure.

Proof. (1) Assume X contains a complemented subspace X_0 isomorphic to l^1. The latter one is, in turn, isometrically isomorphic to the l^1 direct sum $(\oplus_{N\geqslant 2})_{l^1}\ l^1_{N,b}$ of the spaces $l^1_{N,b}$ constructed in Lemma 3.1. Let T_N be an operator on $l^1_{N,b}$ defined by $T_N e_j = \lambda_j e_j$, where $\lambda_j = 1 - q^j$, $0<q<1$, and $1\leqslant j\leqslant N$. Finally, set $T_0 = \oplus_{N\geqslant 2} T_N$.

It follows from Lemmas 2.2 and 3.1 that

$$||R_z(T_0)|| = \sup_{n\geqslant 2} ||R_z(T_N)|| \leqslant \left(\frac{\pi}{2}+1\right)(b+1)|z-1|^{-1} \tag{2}$$

for $|z|>1$, and hence $T_0 \in (TR)$. Moreover, using Lemmas 2.6, 2.11 and 3.1, we get

$$C_{T_N}(n) \geqslant aUB(\mathscr{E}^N) \geqslant c(b+1)N, \tag{3}$$

where $N = [\log(en)]$ and $a, b>0$ are some constants. Therefore, $C_{T_0}(n) \geqslant \sup_{N\geqslant 2} C_{T_N}(n) \geqslant c(b+1)\log(en)$ for every $n\geqslant 1$. It implies also the needed lower estimate for $C(T_0)$, $C_(T_0) \geqslant const(b+1)$.

It remains to set $T = (V^{-1}T_0V)P$, where P is a bounded projection onto X_0, and $V : X_0 \to (\oplus_{N\geqslant 2})_{l^1} l^1_{N,b}$ is an isometric isomorphism mentioned above. Since $X = X_0 \dotplus Ker\ P$ and $T|_{X_0} = V^{-1}T_0V$, $T|_{Ker\ P} = 0$, the result follows.

In the case when X contains a complemented subspace isomorphic to c_0, we can use the dual bases from Lemma 3.1(3).

(2). The same reasoning as in (1) but with the use of Lemma 3.2 instead of Lemma 3.1. $\square$

Passing to constants $C(X,n)$ we need to use more Banach space geometry. Let us recall [LT79] that a Banach space is of *type p* if there is a constant C such that, for any finite sequence $(x_j)_1^n \subset X$, the following inequality holds:

$$\left|\left|\sum\nolimits_{j=1}^n \varepsilon_j x_j\right|\right|_{L^2(0,1;X)} \leqslant C\left(\sum\nolimits_{j=1}^n ||x_j||^p\right)^{\frac{1}{p}},$$

where ε_j are Rademacher functions. In particular, every super-reflexive space, in the sense of James [Jam72] (i.e. a space whose norm is equivalent to a uniformly convex norm), is of type >1. By a deep result of Pisier [Pis82], a Banach space X having type >1 contains the spaces l^2_n, $n\geqslant 1$, *uniformly and uniformly complemented*, i.e. there exists a sequence of subspaces $X_n \subset X$ such that each X_n is isomorphic to the Euclidean space l^2_n, $V_n : X_n \to l^2_n$, with $\sup_{n\geqslant 1} ||V_n||\,||V_N^{-1}|| < \infty$, and there exists a projection $P_n : X \to X$ with range X_n and such that $\sup_{n\geqslant 1} ||P_n|| < \infty$. It is also known that a space X of type 1 *contains the spaces* l^1_n *uniformly* (but probably not uniformly complemented).

We say that a *space X is ST* if it contains the spaces l^1_n or l^∞_n uniformly and uniformly complemented.

Theorem 3.4. (1) *Let X be a ST space. Then*

$$c_1 \log(en) \leqslant C(X,n) \leqslant c_2 \log(en)$$

for every $n \geqslant 1$, where $c_1 = c_1(X) > 0$ and $c_2 > 0$ is an absolute constant. Moreover, for spaces X described in (1) *of Lemma* 3.3, *there exists a Ritt operator $T : X \to X$ such that*

$$c_1 \log(en) \leqslant \frac{C_T(n)}{C(T)} \leqslant c_2 \log(en)$$

for every $n \geqslant 1$.

(2) *Let X be a Banach space of type >1. Then*

$$c_1 \frac{\log(en)}{(\log\log(en))^{\frac{3}{2}}} \leqslant C(X,n) \leqslant c_2 \log(en)$$

for every $n \geqslant 1$, where $c_1 = c_1(X) > 0$ and $c_2 > 0$ is an absolute constant. Moreover, for spaces X described in (2) *of Lemma* 3.3, *there exists a Ritt operator $T : X \to X$ such that*

$$c_1 \frac{\log(en)}{(\log\log(en))^{\frac{3}{2}}} \leqslant \frac{C_T(n)}{C(T)} \leqslant c_2 \log(en)$$

for every $n \geqslant 1$.

(3) *For an arbitrary Banach space X, we have*

$$c_3 \frac{(\log(en))^{\frac{1}{2}}}{(\log\log(en))^{\frac{3}{2}}} \leqslant C(X,n) \leqslant c_2 \log(en),$$

where c_2, $c_3 > 0$ are absolute constants.

Proof. The right-hand side inequalities in (1)–(3) are, of course, direct consequences of Theorem 2.1. Let us consider the left ones.

(1) Let $(X_N)_{N \geqslant 1}$ be a sequence of subspaces of X, $V_N : X_N \to l^1_{N,b}$ (see Lemma 3.3) isomorphisms and $P_N : X \to X_N$ projections such that $S_1 = \sup_{N \geqslant 1} ||V_N||\,||V_n^{-1}|| < \infty$ and $S_2 = \sup_{N \geqslant 1} ||P_N|| < \infty$. Proceeding as in Lemma 3.3 we get an operator $T_N : l^1_{N,b} \to l^1_{N,b}$ satisfying inequalities (2) and (3). Next, as in Lemma 3.3 we set $T_{N,0} = (V_N^{-1} T_N V_N) P_N$ and obtain

$$C_{T_{N,0}}(n) \geqslant \frac{\left(||V_N||\,||V_N^{-1}||\right)^{-1}}{||P_N||} C_{T_N}(n) \geqslant \frac{1}{S_1 S_2} C_{T_N}(n) \geqslant \alpha \log(en),$$

where $N = [\log(en)]$. Similarly,

$$C_{T_{N,0}}(n) \leqslant ||V_N|| \cdot ||V_N^{-1}|| \cdot ||P_N|| \cdot C(T_N) \leqslant S_1 S_2 \left(\frac{\pi}{2}+1\right)(b+1) = \beta.$$

Therefore, $C(X,n) \geqslant \frac{C_{T_{N,0}}(n)}{C(T_{N,0})} \geqslant c_1 \log(en)$ for $N = [\log(en)]$, and the result follows.

The case where X_N are isomorphic to l_N^∞ is similar.

(2) Here, we use Pisier's theorem mentioned above: there exist subspaces $X_N \subset X$, $N = 1, 2, \ldots,$ isomorphims $V_N : X_N \to l_N^2$ and projections $P_N : X \to X_N$ such that $\sup_{N \geqslant 1} ||V_N|| \cdot ||V_N^{-1}|| < \infty$ and $\sup_{N \geqslant 1} ||P_N|| < \infty$. The rest of the proof is the same as in point (1) above but instead of Lemma 3.3 we use Lemma 3.2(2). By the same construction as above, this gives, for every $B > 1$, an operator $T_N : l_N^2 \to l_N^2$ such that $C(T_N) \leqslant \left(\frac{\pi}{2}+1\right)B$ and

$$C_{T_N}(n) \geqslant a \cdot UB(\mathscr{E}^N) \geqslant a \cdot \frac{N^{1-\frac{1}{B}}}{6\sqrt{B}},$$

where $N = [\log(en)]$ and $a > 0$ is an absolute constant. Passing to $T_{N,0} = (V_N^{-1} T_N V_N) P_N$ we obtain as above

$$C_{T_{N,0}}(n) \geqslant \alpha \frac{N^{1-\frac{1}{B}}}{6\sqrt{B}} \quad \text{and} \quad C(T_{N,0}) \leqslant \beta,$$

and hence

$$\begin{aligned} C(X,n) &= \sup\left\{\frac{C_T(n)}{C(T)} : T \in L(X)\right\} \\ &\geqslant \sup\left\{\frac{\beta}{\alpha} B^{-\frac{3}{2}} (\log(en))^{1-\frac{1}{B}} : B > 1\right\} = c_1 \frac{\log(en)}{(\log\log(en))^{\frac{3}{2}}}. \end{aligned}$$

In order to prove (3), we use Dvoretzky's theorem (see, for example, [LT79]) and, for every $N \geqslant 1$ and $\delta > 0$, find a subspace $X_N \subset X$, which is $(1+\delta)$- isomorphic to the Euclidean space $\mathbb{C}^N$. In other words, there exists an isomorphism $V_N : X_N \to \mathbb{C}^N$ with $||V|| \, ||V^{-1}|| \leqslant 1 + \delta$. Moreover, by the Kadec–Snobar theorem (see [Woj91]), there exists a projection P_N on X having the range $P_N X = X_N$ and $||P_N|| \leqslant \sqrt{N}$. This means that, as above, there exists an operator $T = (V_N^{-1} T_N V_N) P_N : X \to X$, where $T_N : \mathbb{C}^N \to \mathbb{C}^N$ satisfies a finite version of the properties from Lemma 3.3(2), that is, $C_T(n) \geqslant \beta B^{-\frac{1}{2}} (\log(en))^{1-\frac{1}{B}}$ for $n \geqslant 1$ satisfying $a \log n \leqslant N$, where $a > 0$ is an absolute constant and, as in Lemma 3.3, $C(T) \leqslant \alpha B$, B being an arbitrary constant $B > 1$. Therefore,

$$\frac{C_{T_N}(n)}{C(T_N)} \geqslant \frac{\beta}{\alpha} B^{-\frac{3}{2}} (\log(en))^{1-\frac{1}{B}},$$

where $a \log n \leqslant N$. It implies that $C(T) \leqslant (1+\delta) \cdot ||P_N|| \cdot C(T_N)$ and

$$\begin{aligned} C_T(n) &= \sup\{||p(T)|| : ||p||_\infty \leqslant 1,\ \deg(p) \leqslant n\} \\ &\geqslant \sup\{||p(T_N)|_{X_N}|| : .||p||_\infty \leqslant 1,\ \deg(p) \leqslant n\} = C_{T_N}(n). \end{aligned}$$

Thus, $\frac{C_T(n)}{C(T)} \geqslant \frac{\beta}{\alpha} B^{-\frac{3}{2}} \frac{(\log(en))^{1-\frac{1}{B}}}{(1+\delta)||P_N||}$. Now, as above, taking the sup over all T's of the above type, we get the needed lower bound for $C(X,n)$. □

We finish with some more comments on lower and upper bounds for constants $C_T(n)$ and $C(X,n)$. First, it is worth mentioning explicitly that in order to get an estimate $C(X,n) \geqslant a \log(en)$, $n \geqslant 1$, by using the approach of Section 2, we need to possess a family of (finite) bases $\mathscr{E}_N$ depending on $\varepsilon > 0$ and $N \geqslant 1$ such that

$$UB(\mathscr{E}_N) \geqslant c \cdot B(\mathscr{E}_N) \cdot N^{1-\varepsilon}$$

and *span*$(\mathscr{E}_N)$ is complemented with a projection norm bounded independently of N and ε. For spaces X containing complemented subspaces isomorphic to a Hilbert space, the problem (of course) reduces to the case of a Hilbert space, where it is still open. (Let us mention that for many spaces X the existence of a subspace isomorphic to a Hilbert space already implies the existence of such a complemented subspace, see Dilworth [Dil85]).

Secondly, if we restrict ourselves to bases $\mathscr{E}_N$ with basis constant $B(\mathscr{E}_N)$ fixed in advance, i.e. $B(\mathscr{E}_N) \leqslant B$ (and hence, to operators $T : X \to X$ with a fixed Ritt constant $C(T) \leqslant \left(\frac{\pi}{2} + 1\right)B$), then, using the construction of Section 2, for many *good* Banach spaces X it is impossible to overcome the growth $C_T(n) \geqslant c_1 C(T)(\log(en))^{1-\varepsilon}$ because of the following result of James [Jam72] and Gurariĭ and Gurariĭ [GG71] giving a generalization of the result of McCarthy and Schwartz [MS65]. Namely, it follows from [GG71,Jam72] that for every super-reflexive (in James' sense) Banach space X (and, in particular, for every uniformly convex X) and for every $B > 1$ there exists $\varepsilon = \varepsilon(X,B) > 0$ such that

$$UB(\mathscr{E}_N) \leqslant c \cdot B \cdot N^{1-\varepsilon}$$

for every basic sequence $\mathscr{E}_N$ in X having $B(\mathscr{E}_N) \leqslant B$, where $c > 0$ is an absolute constant.

Therefore, in order to find the true asymptotic behavior of $C(X,n)$ as $n \to \infty$, one has to look *either* for upper estimates for $C_T(n)$ adapted to a specific Banach space X, which would be better than those of Section 1, *or* for operators T different from McCarthy–Schwartz–Markus operators of Section 2.

Acknowledgments

The author is grateful to Prof. T. Ransford for valuable discussions and helpful comments. She is also indebted to Prof. N. Nikolski for numerous suggestions (especially on the free interpolation problem), and Prof. G. Pisier for very helpful explanations on Banach space geometry.

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